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# A half-discrete Hardy-Hilbert-type inequality related to hyperbolic secant function

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## Abstract

By applying weight functions and technique of real analysis, a half-discrete Hardy-Hilbert-type inequality related to the kernel of hyperbolic secant function and a best possible constant factor are given. The equivalent forms, the operator expressions with the norm, the reverses, and some particular cases are also considered.

**MSC:** 26D15; 47A07; 37A10**Keywords:** Hardy-Hilbert-type inequality; weight function; equivalent form; reverse; operator

## 1 Introduction

If  $p > 1$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $f(x), g(y) \geq 0$ ,  $f \in L^p(\mathbb{R}_+)$ ,  $g \in L^q(\mathbb{R}_+)$ ,  $\|f\|_p = (\int_0^\infty f^p(x) dx)^{\frac{1}{p}} > 0$ , and  $\|g\|_q > 0$ , then we have the following Hardy-Hilbert integral inequality [1]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \quad (1)$$

where, the constant factor  $\frac{\pi}{\sin(\pi/p)}$  is the best possible. If  $a_m, b_n \geq 0$ ,  $a = \{a_m\}_{m=1}^\infty \in l^p$ ,  $b = \{b_n\}_{n=1}^\infty \in l^q$ ,  $\|a\|_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0$ , and  $\|b\|_q > 0$ , then we have the following Hardy-Hilbert's inequality with the same best constant  $\frac{\pi}{\sin(\pi/p)}$  [1]:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \quad (2)$$

Inequalities (1) and (2) are important in analysis and its applications (see [1–5]).

Suppose that  $\mu_i, \nu_j > 0$  ( $i, j \in \mathbb{N} = \{1, 2, \dots\}$ ),

$$U_m := \sum_{i=1}^m \mu_i, \quad V_n := \sum_{j=1}^n \nu_j \quad (m, n \in \mathbb{N}). \quad (3)$$

Then we have the following inequality ([1], Theorem 321):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\mu_m^{1/q} \nu_n^{1/p} a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \quad (4)$$

Replacing  $\mu_m^{1/q} a_m$  and  $\nu_n^{1/p} b_n$  by  $a_m$  and  $b_n$  in (4), respectively, we obtain an equivalent form of (4):

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left( \sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \frac{b_n^q}{\nu_n^{q-1}} \right)^{\frac{1}{q}}. \quad (5)$$

For  $\mu_i = \nu_j = 1$  ( $i, j \in \mathbf{N}$ ), both (4) and (5) reduce to (2). We call (4) and (5) Hardy-Hilbert-type inequalities.

**Note** The authors of [1] did not prove that (4) is valid with the best possible constant factor.

In 1998, by introducing an independent parameter  $\lambda \in (0, 1]$  Yang [6] gave an extension of (1) with the kernel  $\frac{1}{(x+y)^\lambda}$  for  $p = q = 2$ . Later, Yang [5] refined [6] by giving extensions of (1) and (2) as follows.

Assuming that  $\lambda_1, \lambda_2 \in \mathbf{R}$ ,  $\lambda_1 + \lambda_2 = \lambda$ ,  $k_\lambda(x, y)$  is a nonnegative homogeneous function of degree  $-\lambda$  with  $k(\lambda_1) = \int_0^\infty k_\lambda(t, 1) t^{\lambda_1-1} dt \in \mathbf{R}_+$ ,  $\phi(x) = x^{p(1-\lambda_1)-1}$ ,  $\psi(x) = x^{q(1-\lambda_2)-1}$ ,  $f(x), g(y) \geq 0$ ,

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; \|f\|_{p,\phi} := \left\{ \int_0^\infty \phi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$ ,  $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$ , we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy < k(\lambda_1) \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad (6)$$

where the constant factor  $k(\lambda_1)$  is the best possible. Moreover, if  $k_\lambda(x, y)$  keeps finite and  $k_\lambda(x, y) x^{\lambda_1-1} (k_\lambda(x, y) y^{\lambda_2-1})$  is decreasing with respect to  $x > 0$  ( $y > 0$ ), then for  $a_m, b_n \geq 0$ ,

$$a \in l_{p,\phi} = \left\{ a; \|a\|_{p,\phi} := \left( \sum_{n=1}^{\infty} \phi(n) |a_n|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$ ,  $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$ , we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} k_\lambda(m, n) a_m b_n < k(\lambda_1) \|a\|_{p,\phi} \|b\|_{q,\psi}, \quad (7)$$

where the constant factor  $k(\lambda_1)$  is still the best possible.

For  $0 < \lambda_1, \lambda_2 \leq 1$  such that  $\lambda_1 + \lambda_2 = \lambda$ , we set

$$k_\lambda(x, y) = \frac{1}{(x+y)^\lambda} \quad ((x, y) \in \mathbf{R}_+^2).$$

Then by (7) we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1, \lambda_2) \|a\|_{p,\phi} \|b\|_{q,\psi}, \quad (8)$$

where the constant  $B(\lambda_1, \lambda_2)$  is the best possible, and

$$B(u, v) = \int_0^{\infty} \frac{1}{(1+t)^{u+v}} t^{u-1} dt \quad (u, v > 0)$$

is the beta function. Clearly, for  $\lambda = 1$ ,  $\lambda_1 = \frac{1}{q}$ ,  $\lambda_2 = \frac{1}{p}$ , inequality (8) reduces to (2).

In 2015, by adding some conditions, Yang [7] gave an extension of (8) and (5) as follows:

$$\begin{aligned} & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(U_m + V_n)^{\lambda}} \\ & < B(\lambda_1, \lambda_2) \left( \sum_{m=1}^{\infty} \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} \frac{V_n^{q(1-\lambda_2)-1} b_n^q}{\nu_n^{q-1}} \right)^{\frac{1}{q}}, \end{aligned} \quad (9)$$

where the constant  $B(\lambda_1, \lambda_2)$  is still the best possible.

Some other results including multidimensional Hilbert-type inequalities are provided by [8–25].

About the topic of half-discrete Hilbert-type inequalities with inhomogeneous kernels, Hardy *et al.* provided a few results in Theorem 351 of [1], but they did not prove that the constant factors are the best possible. However, Yang [26] gave a result with the kernel  $\frac{1}{(1+nx)^{\lambda}}$  by introducing a variable and proved that the constant factor is the best possible. In 2011, Yang [27] gave the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor  $B(\lambda_1, \lambda_2)$ :

$$\int_0^{\infty} f(x) \left[ \sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} \right] dx < B(\lambda_1, \lambda_2) \|f\|_{p,\phi} \|a\|_{q,\psi}, \quad (10)$$

where,  $\lambda_1 > 0$ ,  $0 < \lambda_2 \leq 1$ ,  $\lambda_1 + \lambda_2 = \lambda$ . Zhong and Yang [17, 28–33] investigated several half-discrete Hilbert-type inequalities with particular kernels. Applying weight functions, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree  $-\lambda \in \mathbf{R}$  with the best constant factor  $k(\lambda_1)$  is obtained as follows:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} k_{\lambda}(x, n) a_n dx < k(\lambda_1) \|f\|_{p,\phi} \|a\|_{q,\psi}, \quad (11)$$

which is an extension of (10) (*cf.* [34]). At the same time, a half-discrete Hilbert-type inequality with a general inhomogeneous kernel and the best constant factor is given by Yang [35]. In 2012–2014, Yang *et al.* published three books [36, 37] and [38] for building the theory of half-discrete Hilbert-type inequalities.

In this paper, by applying weight coefficients and technique of real analysis, a half-discrete Hardy-Hilbert-type inequality related to the kernel of hyperbolic secant function and the best possible constant factor is given, which is an extension of (11) for  $\lambda = 0$  and a particular kernel. The equivalent forms, the operator expressions with the norm, the reverses, and some particular cases are also considered.

## 2 Some lemmas

In the following, we make appointment that  $\mu_i, v_j > 0$  ( $i, j \in \mathbb{N}$ ),  $U_m$  and  $V_n$  are defined by (3),  $\mu(t)$  is a positive continuous function in  $\mathbf{R}_+ = (0, \infty)$ ,

$$U(x) := \int_0^x \mu(t) dt < \infty \quad (x \in [0, \infty)),$$

$v(t) := v_n, t \in (n-1, n]$  ( $n \in \mathbb{N}$ ), and

$$V(y) := \int_0^y v(t) dt \quad (y \in [0, \infty)),$$

$p \neq 0, 1, \frac{1}{p} + \frac{1}{q} = 1, \delta \in \{-1, 1\}, f(x), a_n \geq 0$  ( $x \in \mathbf{R}_+, n \in \mathbb{N}$ ),  $\|f\|_{p, \Phi_\delta} = (\int_0^\infty \Phi_\delta(x) f^p(x) dx)^{\frac{1}{p}}$ ,  $\|a\|_{q, \Psi} = (\sum_{n=1}^\infty \Psi(n) b_n^q)^{\frac{1}{q}}$ , where

$$\Phi_\delta(x) := \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)}, \quad \Psi(n) := \frac{V_n^{q(1-\sigma)-1}}{v_n^{q-1}} \quad (x \in \mathbf{R}_+, n \in \mathbb{N}).$$

**Example 1** For  $\rho, \gamma, \sigma > 0, \alpha > -\rho$ ,  $\sec h(u) = \frac{2}{e^u + e^{-u}}$  ( $u > 0$ ) is called the hyperbolic secant function (cf. [39]), we set  $h(t) = \frac{\sec h(\rho t^\gamma)}{e^{\alpha t^\gamma}}$  ( $t \in \mathbf{R}_+$ ).

(i) Setting  $u = \rho t^\gamma$ , we find

$$\begin{aligned} k(\sigma) &:= \int_0^\infty \frac{\sec h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\ &= \frac{1}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{\sec h(u)}{e^{\frac{\alpha}{\rho} u}} u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-\frac{\alpha}{\rho} u} u^{\frac{\sigma}{\gamma}-1}}{e^u + e^{-u}} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \frac{e^{-(\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1}}{1 + e^{-2u}} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \sum_{k=0}^\infty (-1)^k e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \int_0^\infty \sum_{k=0}^\infty [e^{-(4k+\frac{\alpha}{\rho}+1)u} - e^{-(4k+2+\frac{\alpha}{\rho}+1)u}] u^{\frac{\sigma}{\gamma}-1} du. \end{aligned}$$

By the Lebesgue term-by-term theorem (see [39]), setting  $v = (2k + \frac{\alpha}{\rho} + 1)u$ , we have

$$\begin{aligned} k(\sigma) &= \int_0^\infty \frac{\sec h(\rho t^\gamma)}{e^{\alpha t^\gamma}} t^{\sigma-1} dt \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \int_0^\infty [e^{-(4k+\frac{\alpha}{\rho}+1)u} - e^{-(4k+2+\frac{\alpha}{\rho}+1)u}] u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty (-1)^k \int_0^\infty e^{-(2k+\frac{\alpha}{\rho}+1)u} u^{\frac{\sigma}{\gamma}-1} du \\ &= \frac{2}{\gamma \rho^{\sigma/\gamma}} \sum_{k=0}^\infty \frac{(-1)^k}{(2k + \frac{\alpha}{\rho} + 1)^{\sigma/\gamma}} \int_0^\infty e^{-v} v^{\frac{\sigma}{\gamma}-1} dv \end{aligned}$$

$$\begin{aligned} &= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + \frac{\alpha+\rho}{2\rho})^{\sigma/\gamma}} \\ &= \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \xi\left(\frac{\sigma}{\gamma}, \frac{\alpha+\rho}{2\rho}\right) \in \mathbf{R}_+, \end{aligned} \quad (12)$$

where  $\xi(s, a) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s}$  ( $s, a > 0$ ) and

$$\Gamma(y) := \int_0^{\infty} e^{-v} v^{y-1} dv \quad (y > 0)$$

is the gamma function (see [40]).

In particular, for  $\alpha = \rho > 0$  and  $\gamma = \sigma$ , we have  $h(t) = \frac{\sec h(\rho t^\sigma)}{e^{\rho t^\sigma}}$  and  $k(\sigma) = \frac{\ln 2}{\sigma\rho}$ ; for  $\alpha = 0$  and  $\gamma = \sigma$ , we find  $h(t) = \sec h(\rho t^\sigma)$  and  $k(\sigma) = \frac{\pi}{2\sigma\rho}$ .

(ii) We have  $\frac{1}{e^u + e^{-u}} > 0$  and  $(\frac{1}{e^u + e^{-u}})' = -\frac{e^u - e^{-u}}{(e^u + e^{-u})^2} < 0$  for  $u > 0$ . If  $g(u) > 0$  and  $g'(u) < 0$ , then for  $\gamma > 0$ ,  $g(\rho t^\gamma) > 0$ ,  $\frac{d}{dt}g(\rho t^\gamma) = \rho\gamma t^{\gamma-1}g'(\rho t^\gamma) < 0$ ; for  $y \in (n-1, n)$ ,  $g(V(y)) > 0$ ,  $\frac{d}{dy}g(V(y)) = g'(V(y))v_n < 0$  ( $n \in \mathbf{N}$ ).

If  $g_i(u) > 0$  and  $g'_i(u) < 0$  ( $i = 1, 2$ ), then we find for  $u > 0$ ,

$$g_1(u)g_2(u) > 0, \quad (g_1(u)g_2(u))' = g'_1(u)g_2(u) + g_1(u)g'_2(u) < 0.$$

(iii) Therefore, for  $\rho, \gamma, \sigma > 0$ ,  $\alpha > -\rho$  ( $\alpha \geq 0$ ), we have  $h(t) > 0$  and  $h'(t) < 0$  with  $k(\sigma) \in \mathbf{R}_+$ , and then for  $c > 0$  and  $n \in \mathbf{N}$ , adding  $\sigma \leq 1$ , we have

$$h(cV(y))V^{\sigma-1}(y) > 0, \quad \frac{d}{dy}h(cV(y))V^{\sigma-1}(y) < 0 \quad (y \in (n-1, n)).$$

**Lemma 1** *If  $g(t) (> 0)$  is decreasing in  $\mathbf{R}_+$  and strictly decreasing in  $[n_0, \infty)$  ( $n_0 \in \mathbf{N}$ ) and satisfies  $\int_0^{\infty} g(t) dt \in \mathbf{R}_+$ , then we have*

$$\int_1^{\infty} g(t) dt < \sum_{n=1}^{\infty} g(n) < \int_0^{\infty} g(t) dt. \quad (13)$$

*Proof* Since we have

$$\begin{aligned} \int_n^{n+1} g(t) dt &\leq g(n) \leq \int_{n-1}^n g(t) dt \quad (n = 1, \dots, n_0), \\ \int_{n_0+1}^{n_0+2} g(t) dt &< g(n_0+1) < \int_{n_0}^{n_0+1} g(t) dt, \end{aligned}$$

it follows that

$$0 < \int_1^{n_0+2} g(t) dt < \sum_{n=1}^{n_0+1} g(n) < \sum_{n=1}^{n_0+1} \int_{n-1}^n g(t) dt = \int_0^{n_0+1} g(t) dt < \infty.$$

In the same way, we still have

$$0 < \int_{n_0+2}^{\infty} g(t) dt \leq \sum_{n=n_0+2}^{\infty} g(n) \leq \int_{n_0+1}^{\infty} g(t) dt < \infty.$$

Hence, adding these two inequalities, we have (13). □

**Lemma 2** For  $\gamma, \rho > 0$ ,  $\alpha > -\rho$  ( $\alpha \geq 0$ ), and  $0 < \sigma \leq 1$ , define the following weight coefficients:

$$\omega_\delta(\sigma, x) := \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{U^{\delta\sigma}(x)v_n}{V_n^{1-\sigma}}, \quad x \in \mathbf{R}_+, \quad (14)$$

$$\varpi_\delta(\sigma, n) := \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^\sigma \mu(x)}{U^{1-\delta\sigma}(x)} dx, \quad n \in \mathbf{N}. \quad (15)$$

Then, we have the following inequalities:

$$\omega_\delta(\sigma, x) < k(\sigma) \quad (x \in \mathbf{R}_+), \quad (16)$$

$$\varpi_\delta(\sigma, n) \leq k(\sigma) \quad (n \in \mathbf{N}), \quad (17)$$

where  $k(\sigma)$  is defined in (12).

*Proof* Since  $V_n = V(n)$  and  $V'(t) = v_n$  for  $t \in (n-1, n)$ , by Example 1(iii) and the proof of Lemma 1 we have

$$\begin{aligned} & \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{v_n}{V_n^{1-\sigma}} \\ &= \frac{\sec h(\rho(U^\delta(x)V(n))^\gamma)}{e^{\alpha(U^\delta(x)V(n))^\gamma}} \frac{v_n}{V^{1-\sigma}(n)} \\ &< \int_{n-1}^n \frac{\sec h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{V'(t)}{V^{1-\sigma}(t)} dt \quad (n \in \mathbf{N}), \\ \omega_\delta(\sigma, x) &< \sum_{n=1}^{\infty} \int_{n-1}^n \frac{\sec h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)} dt \\ &= \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{V^{1-\sigma}(t)} dt. \end{aligned}$$

Setting  $u = U^\delta(x)V(t)$ , by (12) we find

$$\begin{aligned} \omega_\delta(\sigma, x) &< \int_0^{U^\delta(x)V(\infty)} \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} \frac{U^{\delta\sigma}(x)U^{-\delta}(x)}{(uU^{-\delta}(x))^{1-\sigma}} du \\ &\leq \int_0^\infty \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \\ &= k(\sigma). \end{aligned}$$

Hence, (16) follows.

Setting  $u = V_n U^\delta(x)$  in (15), we find  $du = \delta V_n U^{\delta-1}(x) \mu(x) dx$  and

$$\begin{aligned} \varpi_\delta(\sigma, n) &= \frac{1}{\delta} \int_{V_n U^\delta(0)}^{V_n U^\delta(\infty)} \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} \frac{V_n^\sigma V_n^{-1} (V_n^{-1} u)^{\frac{1}{\delta}-1}}{(V_n^{-1} u)^{\frac{1}{\delta}-\sigma}} du \\ &= \frac{1}{\delta} \int_{V_n U^\delta(0)}^{V_n U^\delta(\infty)} \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du. \end{aligned}$$

If  $\delta = 1$ , then

$$\varpi_1(\sigma, n) = \int_0^{V_n U(\infty)} \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \leq \int_0^\infty \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du;$$

if  $\delta = -1$ , then

$$\varpi_{-1}(\sigma, n) = - \int_\infty^{V_n U^{-1}(\infty)} \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \leq \int_0^\infty \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du.$$

Then by (12) we have (17).  $\square$

**Remark 1** We do not need  $\sigma \leq 1$  to obtain (17). If  $U(\infty) = \infty$ , then we have

$$\varpi_\delta(\sigma, n) = k(\sigma) \quad (n \in \mathbf{N}). \quad (18)$$

For example, set  $\mu(t) = \frac{1}{(1+t)^\beta}$  ( $t > 0$ ;  $0 \leq \beta \leq 1$ ). Then, for  $x \geq 0$ , we find

$$U(x) = \int_0^x \frac{dt}{(1+t)^\beta} = \begin{cases} \frac{(1+x)^{1-\beta}-1}{1-\beta}, & 0 \leq \beta < 1, \\ \ln(1+x), & \beta = 1 \end{cases} < \infty,$$

and  $U(\infty) = \int_0^\infty \frac{dt}{(1+t)^\beta} = \infty$ .

**Lemma 3** If  $\gamma, \rho > 0, \alpha > -\rho$  ( $\alpha \geq 0$ ),  $0 < \sigma \leq 1$ , then there exists  $n_0 \in \mathbf{N}$  such that  $v_n \geq v_{n+1}$  ( $n \in \{n_0, n_0 + 1, \dots\}$ ), and  $V(\infty) = \infty$ . Moreover, then

(i) for  $x \in \mathbf{R}_+$ , we have

$$k(\sigma)(1 - \theta_\delta(\sigma, x)) < \omega_\delta(\sigma, x), \quad (19)$$

where  $\theta_\delta(\sigma, x) = O((U(x))^{\delta\sigma}) \in (0, 1)$ ;

(ii) for any  $b > 0$ , we have

$$\sum_{n=1}^\infty \frac{v_n}{V_n^{1+b}} = \frac{1}{b} \left( \frac{1}{V_{n_0}^b} + bO(1) \right). \quad (20)$$

*Proof* By Example 1(iii) we have

$$\begin{aligned} \omega_\delta(\sigma, x) &= \sum_{n=1}^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{U^{\delta\sigma}(x)v_n}{V_n^{1-\sigma}} \\ &\geq \sum_{n=n_0}^\infty \int_n^{n+1} \frac{\sec h(\rho(U^\delta(x)V(n))^\gamma)}{e^{\alpha(U^\delta(x)V(n))^\gamma}} \frac{U^{\delta\sigma}(x)v_{n+1}}{(V(n))^{1-\sigma}} dt \\ &> \sum_{n=n_0}^\infty \int_n^{n+1} \frac{\sec h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{(V(t))^{1-\sigma}} dt \\ &= \int_{n_0}^\infty \frac{\sec h(\rho(U^\delta(x)V(t))^\gamma)}{e^{\alpha(U^\delta(x)V(t))^\gamma}} \frac{U^{\delta\sigma}(x)V'(t)}{(V(t))^{1-\sigma}} dt. \end{aligned}$$

Setting  $u = U^\delta(x)V(t)$ , in view of  $V(\infty) = \infty$ , by (12) we find

$$\begin{aligned}\omega_\delta(\sigma, x) &> \int_{U^\delta(x)V_{n_0}}^{\infty} \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \\ &= k(\sigma) - \int_0^{U^\delta(x)V_{n_0}} \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du = k(\sigma)(1 - \theta_\delta(\sigma, x)), \\ \theta_\delta(\sigma, x) &:= \frac{1}{k(\sigma)} \int_0^{U^\delta(x)V_{n_0}} \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} u^{\sigma-1} du \in (0, 1).\end{aligned}$$

Since  $F(u) = \frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}}$  is continuous in  $(0, \infty)$  and satisfies  $F(u) \rightarrow 1$  ( $u \rightarrow 0^+$ ),  $F(u) \rightarrow 0$  ( $u \rightarrow \infty$ ), there exists a constant  $L > 0$  such that  $F(u) \leq L$ , namely,  $\frac{\sec h(\rho u^\gamma)}{e^{\alpha u^\gamma}} \leq L$  ( $u \in (0, \infty)$ ). Hence, we find

$$0 < \theta_\delta(\sigma, x) \leq \frac{L}{k(\sigma)} \int_0^{U^\delta(x)V_{n_0}} u^{\sigma-1} du = \frac{L(U^\delta(x)V_{n_0})^\sigma}{k(\sigma)\sigma},$$

and then (19) follows.

For  $b > 0$ , we find

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1+b}} &= \sum_{n=1}^{n_0} \frac{v_n}{V_n^{1+b}} + \sum_{n=n_0+1}^{\infty} \frac{v_n}{V_n^{1+b}(n)} \\ &< \sum_{n=1}^{n_0} \frac{v_n}{V_n^{1+b}} + \sum_{n=n_0+1}^{\infty} \int_{n-1}^n \frac{V'(x)}{V^{1+b}(x)} dx \\ &= \sum_{n=1}^{n_0} \frac{v_n}{V_n^{1+b}} + \int_{n_0}^{\infty} \frac{dV(x)}{V^{1+b}(x)} = \sum_{n=1}^{n_0} \frac{v_n}{V_n^{1+b}} + \frac{1}{bV^b(n_0)} \\ &= \frac{1}{b} \left( \frac{1}{V_{n_0}^b} + b \sum_{n=1}^{n_0} \frac{v_n}{V_n^{1+b}} \right), \\ \sum_{n=1}^{\infty} \frac{v_n}{V_n^{1+b}} &\geq \sum_{n=n_0}^{\infty} \int_n^{n+1} \frac{v_{n+1}}{V^{1+b}(n)} dx > \sum_{n=n_0}^{\infty} \int_n^{n+1} \frac{V'(x) dx}{V^{1+b}(x)} \\ &= \int_{n_0}^{\infty} \frac{dV(x)}{V^{1+b}(x)} = \frac{1}{bV^b(n_0)} = \frac{1}{bV_{n_0}^b}.\end{aligned}$$

Hence, we have (20). □

**Note** For example,  $v_n = \frac{1}{n^\beta}$  ( $n \in \mathbb{N}$ ;  $0 \leq \beta \leq 1$ ) satisfies the conditions of Lemma 3 (for  $n_0 = 1$ ).

### 3 Main results and operator expressions

**Theorem 1** If  $\gamma, \rho > 0$ ,  $\alpha > -\rho$  ( $\alpha \geq 0$ ),  $0 < \sigma \leq 1$ ,  $k(\sigma)$  is defined in by (12), then for  $p > 1$ ,  $0 < \|f\|_{p, \Phi_\delta}, \|a\|_{q, \Psi} < \infty$ , we have the following equivalent inequalities:

$$I := \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n f(x) dx < k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad (21)$$



$$I_1 := \sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right]^p$$

$$< k(\sigma) \|f\|_{p, \Phi_\delta}, \quad (22)$$

$$I_2 := \left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}}$$

$$< k(\sigma) \|a\|_{q, \Psi}. \quad (23)$$

*Proof* By Hölder's inequality with weight (see [41]), we have

$$\begin{aligned} & \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right]^p \\ &= \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \left( \frac{U^{\frac{1-\delta\sigma}{q}}(x)f(x)}{V_n^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)} \right) \left( \frac{V_n^{\frac{1-\sigma}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1-\delta\sigma}{q}}(x)} \right) dx \right]^p \\ &\leq \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \left( \frac{U^{\frac{p(1-\delta\sigma)}{q}}(x)f^p(x)}{V_n^{1-\sigma} \mu^{\frac{p}{q}}(x)} \right) dx \\ &\quad \times \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{(1-\sigma)(p-1)} \mu(x)}{U^{1-\delta\sigma}(x)} dx \right]^{p-1} \\ &= \frac{(\varpi_\delta(\sigma, n))^{p-1}}{V_n^{p\sigma-1} v_n} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_n}{V_n^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx. \end{aligned} \quad (24)$$

In view of (17) and the Lebesgue term-by-term integration theorem (see [42]), we find

$$\begin{aligned} I_1 &\leq (k(\sigma))^{\frac{1}{q}} \left[ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_n}{V_n^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}} \left[ \int_0^{\infty} \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_n}{V_n^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}} \left[ \int_0^{\infty} \omega_\delta(\sigma, x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned} \quad (25)$$

Then by (16) we have (22).

By Hölder's inequality (see [41]) we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \left[ \frac{v_n^{\frac{1}{p}}}{V_n^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right] \left( \frac{V_n^{\frac{1}{p}-\sigma} a_n}{v_n^{\frac{1}{p}}} \right) \\ &\leq I_1 \|a\|_{q, \Psi}. \end{aligned} \quad (26)$$

Then by (22) we have (21). On the other hand, assuming that (21) is valid, we set

$$a_n := \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right]^{p-1}, \quad n \in \mathbb{N}.$$

Then we find  $J_1^p = \|a\|_{q,\Psi}^q$ . If  $J_1 = 0$ , then (22) is trivially valid; if  $J_1 = \infty$ , then (22) keeps impossible. Suppose that  $0 < J_1 < \infty$ . By (21) we have

$$\|a\|_{q,\Psi}^q = J_1^p = I < k(\sigma) \|f\|_{p,\Phi_\delta} \|a\|_{q,\Psi}, \quad \|a\|_{q,\Psi}^{q-1} = J_1 < k(\sigma) \|f\|_{p,\Phi_\delta},$$

and then (22) follows, which is equivalent to (21).

Again by Hölder's inequality with weight we have

$$\begin{aligned} & \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right]^q \\ &= \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \left( \frac{U^{\frac{1-\delta\sigma}{q}}(x)v_n^{\frac{1}{p}}}{V_n^{\frac{1-\sigma}{p}}} \right) \left( \frac{V_n^{\frac{1-\sigma}{p}} a_n}{U^{\frac{1-\delta\sigma}{q}}(x)v_n^{\frac{1}{p}}} \right) \right]^q \\ &\leq \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x)v_n}{V_n^{1-\sigma}} \right]^{q-1} \\ &\quad \times \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{\frac{q(1-\sigma)}{p}}}{U^{1-\delta\sigma}(x)v_n^{q-1}} a_n^q \\ &= \frac{(\omega_\delta(\sigma, x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)v_n^{q-1}} a_n^q. \end{aligned} \quad (27)$$

Then by (16) and the Lebesgue term-by-term integration theorem it follows that

$$\begin{aligned} J_2 &< (k(\sigma))^{\frac{1}{p}} \left\{ \int_0^\infty \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)v_n^{q-1}} a_n^q dx \right\}^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)v_n^{q-1}} a_n^q dx \right\}^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \varpi_\delta(\sigma, n) \frac{V_n^{q(1-\sigma)-1}}{v_n^{q-1}} a_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (28)$$

Then by (17) we have (23).

By Hölder's inequality we have

$$\begin{aligned} I &= \int_0^\infty \left( \frac{U^{\frac{1-\delta\sigma}{q}}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right) \left[ \frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1-\delta\sigma}{q}}(x)} \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right] dx \\ &\leq \|f\|_{p,\Phi_\delta} J_2. \end{aligned} \quad (29)$$

Then by (23) we have (21). On the other hand, assuming that (23) is valid, we set

$$f(x) := \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right]^{q-1}, \quad x \in \mathbf{R}_+.$$

Then we find  $J_2^q = \|f\|_{p,\Phi_\delta}^p$ . If  $J_2 = 0$ , then (23) is trivially valid; if  $J_2 = \infty$ , then (23) keeps impossible. Suppose that  $0 < J_2 < \infty$ . By (21) we have

$$\|f\|_{p,\Phi_\delta}^p = J_2^q = I < k(\sigma) \|f\|_{p,\Phi_\delta} \|a\|_{q,\Psi}, \quad \|f\|_{p,\Phi_\delta}^{p-1} = J_2 < k(\sigma) \|a\|_{q,\Psi},$$

and then (23) follows, which is equivalent to (21).

Therefore, (21), (22), and (23) are equivalent.  $\square$

**Theorem 2** *With the assumptions of Theorem 1, if there exists  $n_0 \in \mathbf{N}$  such that  $v_n \geq v_{n+1}$  ( $n \in \{n_0, n_0 + 1, \dots\}$ ) and  $U(\infty) = V(\infty) = \infty$ , then the constant factor  $k(\sigma)$  in (21), (22), and (23) is the best possible.*

*Proof* For  $\varepsilon \in (0, q\sigma)$ , we set  $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}$  and  $\tilde{f} = \tilde{f}(x)$ ,  $x \in \mathbf{R}_+$ ,  $\tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$ ,

$$\tilde{f}(x) = \begin{cases} U^{\delta(\tilde{\sigma}+\varepsilon)-1}(x)\mu(x), & 0 < x^\delta \leq 1, \\ 0, & x^\delta > 0, \end{cases} \quad (30)$$

$$\tilde{a}_n = V_n^{\tilde{\sigma}-1} v_n = V_n^{\sigma-\frac{\varepsilon}{q}-1} v_n, \quad n \in \mathbf{N}. \quad (31)$$

Then for  $\delta = \pm 1$ , since  $U(\infty) = \infty$ , we find

$$\int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx = \frac{1}{\varepsilon} U^{\delta\varepsilon}(1). \quad (32)$$

By (20), (32), and (19) we obtain

$$\begin{aligned} \|\tilde{f}\|_{p,\Phi_\delta} \|\tilde{a}\|_{q,\Psi} &= \left( \int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\mu(x) dx}{U^{1-\delta\varepsilon}(x)} \right)^{\frac{1}{p}} \left( \sum_{n=1}^\infty \frac{v_n}{V_n^{1+\varepsilon}} \right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} U^{\frac{\delta\varepsilon}{p}}(1) \left( \frac{1}{V_{n_0}^\varepsilon} + \varepsilon O(1) \right)^{\frac{1}{q}}, \end{aligned} \quad (33)$$

$$\begin{aligned} \tilde{I} &:= \int_0^\infty \sum_{n=1}^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \tilde{a}_n \tilde{f}(x) dx \\ &= \int_{\{x>0; 0<x^\delta \leq 1\}} \sum_{n=1}^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{\tilde{\sigma}-1} v_n \mu(x)}{U^{1-\delta(\tilde{\sigma}+\varepsilon)}(x)} dx \\ &= \int_{\{x>0; 0<x^\delta \leq 1\}} \omega_\delta(\tilde{\sigma}, x) \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx \\ &\geq k(\tilde{\sigma}) \int_{\{x>0; 0<x^\delta \leq 1\}} (1 - \theta_\delta(\tilde{\sigma}, x)) \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx \\ &= k(\tilde{\sigma}) \int_{\{x>0; 0<x^\delta \leq 1\}} (1 - O((U(x))^{\delta(\sigma-\frac{\varepsilon}{q})})) \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx \\ &= k(\tilde{\sigma}) \left[ \int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\mu(x)}{U^{1-\delta\varepsilon}(x)} dx - \int_{\{x>0; 0<x^\delta \leq 1\}} O\left(\frac{\mu(x)}{U^{1-\delta(\sigma+\frac{\varepsilon}{p})}(x)}\right) dx \right] \\ &= \frac{1}{\varepsilon} k\left(\sigma - \frac{\varepsilon}{q}\right) (U^{\delta\varepsilon}(1) - \varepsilon O(1)). \end{aligned}$$

If there exists a positive constant  $K \leq k(\sigma)$  such that (21) is valid when replacing  $k(\sigma)$  by  $K$ , then, in particular, by the Lebesgue term-by-term integration theorem we have  $\varepsilon \tilde{I} < \varepsilon K \|\tilde{f}\|_{p, \Phi_\delta} \|\tilde{a}\|_{q, \Psi}$ , namely,

$$k\left(\sigma - \frac{\varepsilon}{q}\right)(U^{\delta\varepsilon}(1) - \varepsilon O(1)) < K \cdot U^{\frac{\delta\varepsilon}{p}}(1) \left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon O(1)\right)^{\frac{1}{q}}.$$

It follows that  $k(\sigma) \leq K$  ( $\varepsilon \rightarrow 0^+$ ). Hence,  $K = k(\sigma)$  is the best possible constant factor of (21).

The constant factor  $k(\sigma)$  in (22) ((23)) is still the best possible. Otherwise, we would reach a contradiction by (26) ((29)) that the constant factor in (21) is not the best possible.  $\square$

For  $p > 1$ , we find  $\Psi^{1-p}(n) = \frac{v_n}{V_n^{1-p\sigma}}$  ( $n \in \mathbf{N}$ ),  $\Phi_\delta^{1-q}(x) = \frac{\mu(x)}{U^{1-q\delta\sigma}(x)}$  ( $x \in \mathbf{R}_+$ ) and define the following real normed spaces:

$$\begin{aligned} L_{p, \Phi_\delta}(\mathbf{R}_+) &= \{f; f = f(x), x \in \mathbf{R}_+, \|f\|_{p, \Phi_\delta} < \infty\}, \\ l_{q, \Psi} &= \{a; a = \{a_n\}_{n=1}^\infty, \|a\|_{q, \Psi} < \infty\}, \\ L_{q, \Phi_\delta^{1-q}}(\mathbf{R}_+) &= \{h; h = h(x), x \in \mathbf{R}_+, \|h\|_{q, \Phi_\delta^{1-q}} < \infty\}, \\ l_{p, \Psi^{1-p}} &= \{c; c = \{c_n\}_{n=1}^\infty, \|c\|_{p, \Psi^{1-p}} < \infty\}. \end{aligned}$$

Assuming that  $f \in L_{p, \Phi_\delta}(\mathbf{R}_+)$  and setting

$$c = \{c_n\}_{n=1}^\infty, \quad c_n := \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx, \quad n \in \mathbf{N},$$

we can rewrite (22) as  $\|c\|_{p, \Psi^{1-p}} < k(\sigma) \|f\|_{p, \Phi_\delta} < \infty$ , namely,  $c \in l_{p, \Psi^{1-p}}$ .

**Definition 1** Define a half-discrete Hardy-Hilbert-type operator  $T_1 : L_{p, \Phi_\delta}(\mathbf{R}_+) \rightarrow l_{p, \Psi^{1-p}}$  as follows: For any  $f \in L_{p, \Phi_\delta}(\mathbf{R}_+)$ , there exists a unique representation  $T_1 f = c \in l_{p, \Psi^{1-p}}$ . We define the formal inner product of  $T_1 f$  and  $a = \{a_n\}_{n=1}^\infty \in l_{q, \Psi}$  as follows:

$$(T_1 f, a) := \sum_{n=1}^\infty \left[ \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right] a_n. \quad (34)$$

Then we can rewrite (21) and (22) as follows:

$$(T_1 f, a) < k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad (35)$$

$$\|T_1 f\|_{p, \Psi^{1-p}} < k(\sigma) \|f\|_{p, \Phi_\delta}. \quad (36)$$

Define the norm of operator  $T_1$  as follows:

$$\|T_1\| := \sup_{f(\neq 0) \in L_{p, \Phi_\delta}(\mathbf{R}_+)} \frac{\|T_1 f\|_{p, \Psi^{1-p}}}{\|f\|_{p, \Phi_\delta}}.$$

Then by (36) it follows that  $\|T_1\| \leq k(\sigma)$ . Since by Theorem 2 the constant factor in (36) is the best possible, we have

$$\|T_1\| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \xi\left(\frac{\sigma}{\gamma}, \frac{\alpha + \rho}{2\rho}\right). \quad (37)$$

Assuming that  $a = \{a_n\}_{n=1}^\infty \in l_{q,\Psi}$  and setting

$$h(x) := \sum_{n=1}^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n, \quad x \in \mathbf{R}_+,$$

we can rewrite (23) as  $\|h\|_{q,\Phi_\delta^{1-q}} < k(\sigma)\|a\|_{q,\Psi} < \infty$ , namely,  $h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$ .

**Definition 2** Define a half-discrete Hardy-Hilbert-type operator  $T_2 : l_{q,\Psi} \rightarrow L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$  as follows: For any  $a = \{a_n\}_{n=1}^\infty \in l_{q,\Psi}$ , there exists a unique representation  $T_2 a = h \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+)$ . We define the formal inner product of  $T_2 a$  and  $f \in L_{p,\Phi_\delta}(\mathbf{R}_+)$  as follows:

$$(T_2 a, f) := \int_0^\infty \left[ \sum_{n=1}^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right] f(x) dx. \quad (38)$$

Then we can rewrite (21) and (23) as follows:

$$(T_2 a, f) < k(\sigma)\|f\|_{p,\Phi_\delta}\|a\|_{q,\Psi}, \quad (39)$$

$$\|T_2 a\|_{q,\Phi_\delta^{1-q}} < k(\sigma)\|a\|_{q,\Psi}. \quad (40)$$

Define the norm of operator  $T_2$  as follows:

$$\|T_2\| := \sup_{a(\neq \theta) \in l_{q,\Psi}} \frac{\|T_2 a\|_{q,\Phi_\delta^{1-q}}}{\|a\|_{q,\Psi}}.$$

Then by (40) we find  $\|T_2\| \leq k(\sigma)$ . Since by Theorem 2 the constant factor in (40) is the best possible, we have

$$\|T_2\| = k(\sigma) = \frac{2\Gamma(\frac{\sigma}{\gamma})}{\gamma(2\rho)^{\sigma/\gamma}} \xi\left(\frac{\sigma}{\gamma}, \frac{\alpha + \rho}{2\rho}\right) = \|T_1\|. \quad (41)$$

#### 4 Some equivalent reverse inequalities

In the following, we also set

$$\tilde{\Phi}_\delta(x) := (1 - \theta_\delta(\sigma, x)) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} \quad (x \in \mathbf{R}_+).$$

For  $0 < p < 1$  or  $p < 0$ , we still use the formal symbols  $\|f\|_{p,\Phi_\delta}$ ,  $\|f\|_{p,\tilde{\Phi}_\delta}$ , and  $\|a\|_{q,\Psi}$ .

**Theorem 3** If  $\gamma, \rho > 0$ ,  $\alpha > -\rho$  ( $\alpha \geq 0$ ),  $0 < \sigma \leq 1$ ,  $k(\sigma)$  is defined in (12), there exists  $n_0 \in \mathbf{N}$  such that  $v_n \geq v_{n+1}$  ( $n \in \{n_0, n_0 + 1, \dots\}$ ), and  $U(\infty) = V(\infty) = \infty$ , then for  $p < 0$ ,  $0 <$

$\|f\|_{p,\Phi_\delta}, \|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $k(\sigma)$ :

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p,\Phi_\delta} \|a\|_{q,\Psi}, \quad (42)$$

$$J_1 = \sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p,\Phi_\delta}, \quad (43)$$

$$J_2 = \left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) \|a\|_{q,\Psi}. \quad (44)$$

*Proof* By the reverse Hölder inequality with weight (see [41]), since  $p < 0$ , similarly as obtaining (24) and (25), we have

$$\begin{aligned} & \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right]^p \\ & \leq \frac{(\varpi_\delta(\sigma, n))^{p-1}}{V_n^{p\sigma-1} v_n} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_n}{V_n^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx. \end{aligned}$$

Then by (18) and the Lebesgue term-by-term integration theorem it follows that

$$\begin{aligned} J_1 & \geq (k(\sigma))^{\frac{1}{q}} \left[ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_n}{V_n^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}} \\ & = (k(\sigma))^{\frac{1}{q}} \left[ \int_0^{\infty} \omega_\delta(\sigma, x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Then by (16) we have (43).

By the reverse Hölder inequality we have

$$\begin{aligned} I & = \sum_{n=1}^{\infty} \left[ \frac{v_n^{\frac{1}{p}}}{V_n^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right] \left( \frac{V_n^{\frac{1}{p}-\sigma} a_n}{v_n^{\frac{1}{p}}} \right) \\ & \geq J_1 \|a\|_{q,\Psi}. \end{aligned} \quad (45)$$

Then by (43) we have (42). On the other hand, assuming that (42) is valid, we set  $a_n$  as in Theorem 1. Then we find  $J_1^p = \|a\|_{q,\Psi}^q$ . If  $J_1 = \infty$ , then (43) is trivially valid; if  $J_1 = 0$ , then (43) keeps impossible. Suppose that  $0 < J_1 < \infty$ . By (42) it follows that

$$\|a\|_{q,\Psi}^q = J_1^p = I > k(\sigma) \|f\|_{p,\Phi_\delta} \|a\|_{q,\Psi},$$

$$\|a\|_{q,\Psi}^{q-1} = J_1 > k(\sigma) \|f\|_{p,\Phi_\delta},$$

and then (43) follows, which is equivalent to (42).

Still by the reverse Hölder inequality with weight, since  $0 < q < 1$ , similarly as obtaining (27) and (28), we have

$$\begin{aligned} & \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right]^q \\ & \geq \frac{(\omega_\delta(\sigma, x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)v_n^{q-1}} a_n^q. \end{aligned}$$

Then by (16) and the Lebesgue term-by-term integration theorem it follows that

$$\begin{aligned} J_2 & > (k(\sigma))^{\frac{1}{p}} \left\{ \int_0^\infty \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{(1-\sigma)(q-1)}\mu(x)}{U^{1-\delta\sigma}(x)v_n^{q-1}} a_n^q dx \right\}^{\frac{1}{q}} \\ & = (k(\sigma))^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \varpi_\delta(\sigma, n) \frac{V_n^{q(1-\sigma)-1}}{v_n^{q-1}} a_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (18) we have (44).

By the reverse Hölder inequality we have

$$\begin{aligned} I & = \int_0^\infty \left( \frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right) \left[ \frac{\mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right] dx \\ & \geq \|f\|_{p, \Phi_\delta} J_2. \end{aligned} \quad (46)$$

Then by (44) we have (42). On the other hand, assuming that (44) is valid, we set  $f(x)$  as in Theorem 1. Then we find  $J_2^q = \|f\|_{p, \Phi_\delta}^p$ . If  $J_2 = \infty$ , then (44) is trivially valid; if  $J_2 = 0$ , then (44) keeps impossible. Suppose that  $0 < J_2 < \infty$ . By (42) it follows that

$$\|f\|_{p, \Phi_\delta}^p = J_2^q = I > k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad \|f\|_{p, \Phi_\delta}^{p-1} = J_2 > k(\sigma) \|a\|_{q, \Psi},$$

and then (44) follows, which is equivalent to (42).

Therefore, inequalities (42), (43), and (44) are equivalent.

For  $\varepsilon \in (0, q\sigma)$ , we set  $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}$  and  $\tilde{f} = \tilde{f}(x)$ ,  $x \in \mathbf{R}_+$ ,  $\tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$ ,

$$\tilde{f}(x) = \begin{cases} U^{\delta(\tilde{\sigma}+\varepsilon)-1}(x)\mu(x), & 0 < x^\delta \leq 1, \\ 0, & x^\delta > 0, \end{cases}$$

$$\tilde{a}_n = \tilde{V}_n^{\tilde{\sigma}-1} v_n = \tilde{V}_n^{\sigma-\frac{\varepsilon}{q}-1} v_n, \quad n \in \mathbf{N}.$$

By (20), (32), and (16) we obtain

$$\begin{aligned} \| \tilde{f} \|_{p, \Phi_\delta} \| \tilde{a} \|_{q, \Psi} & = \frac{1}{\varepsilon} U^{\frac{\delta\varepsilon}{p}}(1) \left( \frac{1}{V_{n_0}^\varepsilon} + \varepsilon O(1) \right)^{\frac{1}{q}}, \\ \tilde{I} & = \sum_{n=1}^{\infty} \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \tilde{a}_n \tilde{f}(x) dx \end{aligned}$$

$$\begin{aligned} &= \int_{\{x>0; 0<x^\delta \leq 1\}} \omega_\delta(\tilde{\sigma}, x) \frac{\mu(x)}{U^{1-\delta\epsilon}(x)} dx \\ &\leq k(\tilde{\sigma}) \int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\mu(x)}{U^{1-\delta\epsilon}(x)} dx = \frac{1}{\epsilon} k\left(\sigma - \frac{\epsilon}{q}\right) U^{\delta\epsilon}(1). \end{aligned}$$

If there exists a positive constant  $K \geq k(\sigma)$  such that (42) is valid when replacing  $k(\sigma)$  by  $K$ , then, in particular, we have  $\epsilon \tilde{I} > \epsilon K \|\tilde{f}\|_{p, \Phi_\delta} \|\tilde{a}\|_{q, \Psi}$ , namely,

$$k\left(\sigma - \frac{\epsilon}{q}\right) U^{\delta\epsilon}(1) > K \cdot U^{\frac{\delta\epsilon}{p}}(1) \left(\frac{1}{V_{n_0}^\epsilon} + \epsilon O(1)\right)^{\frac{1}{q}}.$$

It follows that  $k(\sigma) \geq K$  ( $\epsilon \rightarrow 0^+$ ). Hence,  $K = k(\sigma)$  is the best possible constant factor of (42).

The constant factor  $k(\sigma)$  in (43) ((44)) is still the best possible. Otherwise, we would reach a contradiction by (45) ((46)) that the constant factor in (42) is not the best possible.  $\square$

**Theorem 4** *With the assumptions of Theorem 3, if  $0 < p < 1$ ,  $0 < \|f\|_{p, \Phi_\delta}$ , and  $\|a\|_{q, \Psi} < \infty$ , then we have the following equivalent inequalities with the best possible constant factor  $k(\sigma)$ :*

$$I = \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x) V_n)^\gamma)}{e^{\alpha(U^\delta(x) V_n)^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad (47)$$

$$J_1 = \sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x) V_n)^\gamma)}{e^{\alpha(U^\delta(x) V_n)^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p, \Phi_\delta}^p, \quad (48)$$

$$\begin{aligned} J &:= \left\{ \int_0^{\infty} \frac{(1 - \theta_\delta(\sigma, x))^{1-q} \mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x) V_n)^\gamma)}{e^{\alpha(U^\delta(x) V_n)^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} \\ &> k(\sigma) \|a\|_{q, \Psi}. \end{aligned} \quad (49)$$

*Proof* By the reverse Hölder inequality with weight, since  $0 < p < 1$ , similarly as obtaining (24) and (25), we have

$$\begin{aligned} &\left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x) V_n)^\gamma)}{e^{\alpha(U^\delta(x) V_n)^\gamma}} f(x) dx \right]^p \\ &\geq \frac{(\varpi_\delta(\sigma, n))^{p-1}}{V_n^{p\sigma-1} v_n} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x) V_n)^\gamma)}{e^{\alpha(U^\delta(x) V_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_n}{V_n^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx. \end{aligned}$$

In view of (18) and the Lebesgue term-by-term integration theorem, we find

$$\begin{aligned} J_1 &\geq (k(\sigma))^{\frac{1}{q}} \left[ \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x) V_n)^\gamma)}{e^{\alpha(U^\delta(x) V_n)^\gamma}} \frac{U^{(1-\delta\sigma)(p-1)}(x) v_n}{V_n^{1-\sigma} \mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}} \\ &= (k(\sigma))^{\frac{1}{q}} \left[ \int_0^{\infty} \omega_\delta(\sigma, x) \frac{U^{p(1-\delta\sigma)-1}(x)}{\mu^{p-1}(x)} f^p(x) dx \right]^{\frac{1}{p}}. \end{aligned}$$

Then by (19) we have (48).



By the reverse Hölder inequality we have

$$\begin{aligned} I &= \sum_{n=1}^{\infty} \left[ \frac{v_n^{\frac{1}{p}}}{V_n^{\frac{1}{p}-\sigma}} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} f(x) dx \right] \left( \frac{V_n^{\frac{1}{p}-\sigma} a_n}{v_n^{\frac{1}{p}}} \right) \\ &\geq J_1 \|a\|_{q,\Psi}. \end{aligned} \quad (50)$$

Then by (48) we have (47). On the other hand, assuming that (47) is valid, we set  $a_n$  as in Theorem 1. Then we find  $J_1^p = \|a\|_{q,\Psi}^q$ . If  $J_1 = \infty$ , then (48) is trivially valid; if  $J_1 = 0$ , then (48) keeps impossible. Suppose that  $0 < J_1 < \infty$ . By (47) it follows that

$$\|a\|_{q,\Psi}^q = J_1^p = I > k(\sigma) \|f\|_{p,\tilde{\Phi}_\delta} \|a\|_{q,\Psi}, \quad \|a\|_{q,\Psi}^{q-1} = J_1 > k(\sigma) \|f\|_{p,\tilde{\Phi}_\delta},$$

and then (48) follows, which is equivalent to (47).

Again by the reverse Hölder inequality with weight, since  $q < 0$ , we have

$$\begin{aligned} &\left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right]^q \\ &\leq \frac{(\omega_\delta(\sigma, x))^{q-1}}{U^{q\delta\sigma-1}(x)\mu(x)} \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{(1-\sigma)(q-1)} \mu(x)}{U^{1-\delta\sigma}(x)v_n^{q-1}} a_n^q. \end{aligned}$$

Then by (19) and the Lebesgue term-by-term integration theorem it follows that

$$\begin{aligned} J &> (k(\sigma))^{\frac{1}{p}} \left\{ \int_0^{\infty} \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{(1-\sigma)(q-1)} \mu(x)}{U^{1-\delta\sigma}(x)v_n^{q-1}} a_n^q dx \right\}^{\frac{1}{q}} \\ &= (k(\sigma))^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \varpi_\delta(\sigma, n) \frac{V_n^{q(1-\sigma)-1}}{v_n^{q-1}} a_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (18) we have (49).

By the reverse Hölder inequality we have

$$\begin{aligned} I &= \int_0^{\infty} \left[ (1 - \theta_\delta(\sigma, x))^{\frac{1}{p}} \frac{U^{\frac{1}{q}-\delta\sigma}(x)}{\mu^{\frac{1}{q}}(x)} f(x) \right] \\ &\quad \times \left[ \frac{(1 - \theta_\delta(\sigma, x))^{\frac{-1}{p}} \mu^{\frac{1}{q}}(x)}{U^{\frac{1}{q}-\delta\sigma}(x)} \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} a_n \right] dx \\ &\geq \|f\|_{p,\tilde{\Phi}_\delta} J. \end{aligned} \quad (51)$$

Then by (49) we have (47). On the other hand, assuming that (47) is valid, we set  $f(x)$  as in Theorem 1. Then we find  $J^q = \|f\|_{p,\tilde{\Phi}_\delta}^p$ . If  $J = \infty$ , then (49) is trivially valid; if  $J = 0$ , then (49) keeps impossible. Suppose that  $0 < J < \infty$ . By (47) it follows that

$$\|f\|_{p,\tilde{\Phi}_\delta}^p = J^q = I > k(\sigma) \|f\|_{p,\tilde{\Phi}_\delta} \|a\|_{q,\Psi}, \quad \|f\|_{p,\tilde{\Phi}_\delta}^{p-1} = J > k(\sigma) \|a\|_{q,\Psi},$$

and then (49) follows, which is equivalent to (47).

Therefore, inequalities (47), (48), and (49) are equivalent.

For  $\varepsilon \in (0, p\sigma)$ , we set  $\tilde{\sigma} = \sigma + \frac{\varepsilon}{p}$  and  $\tilde{f} = \tilde{f}(x)$ ,  $x \in \mathbf{R}_+$ ,  $\tilde{a} = \{\tilde{a}_n\}_{n=1}^\infty$ ,

$$\tilde{f}(x) = \begin{cases} U^{\delta\tilde{\sigma}-1}(x)\mu(x), & 0 < x^\delta \leq 1, \\ 0, & x^\delta > 0, \end{cases}$$

$$\tilde{a}_n = \tilde{V}_n^{\tilde{\sigma}-\varepsilon-1}v_n = \tilde{V}_n^{\sigma-\frac{\varepsilon}{q}-1}v_n, \quad n \in \mathbf{N}.$$

By (19), (20), and (32) we obtain

$$\begin{aligned} \|\tilde{f}\|_{p,\tilde{\Phi}_\delta} \|\tilde{a}\|_{q,\Psi} &= \left[ \int_{\{x>0; 0<x^\delta \leq 1\}} (1 - O((U(x))^{\delta\sigma})) \frac{\mu(x)dx}{U^{1-\delta\varepsilon}(x)} \right]^{\frac{1}{p}} \left( \sum_{n=1}^\infty \frac{v_n}{V_n^{1+\varepsilon}} \right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} (U^{\delta\varepsilon}(1) - \varepsilon O(1))^{\frac{1}{p}} \left( \frac{1}{V_{n_0}^\varepsilon} + \varepsilon O(1) \right)^{\frac{1}{q}}, \\ \tilde{I} &= \sum_{n=1}^\infty \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \tilde{a}_n \tilde{f}(x) dx \\ &= \sum_{n=1}^\infty \left( \int_{\{x>0; 0<x^\delta \leq 1\}} \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{\tilde{\sigma}} \mu(x)}{U^{1-\delta\tilde{\sigma}}(x)} dx \right) \frac{v_n}{V_n^{1+\varepsilon}} \\ &\leq \sum_{n=1}^\infty \left( \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\gamma)}{e^{\alpha(U^\delta(x)V_n)^\gamma}} \frac{V_n^{\tilde{\sigma}} \mu(x)}{U^{1-\delta\tilde{\sigma}}(x)} dx \right) \frac{v_n}{V_n^{1+\varepsilon}} \\ &= \sum_{n=1}^\infty \varpi_\delta(\tilde{\sigma}, n) \frac{v_n}{V_n^{1+\varepsilon}} = k(\tilde{\sigma}) \sum_{n=1}^\infty \frac{v_n}{V_n^{1+\varepsilon}} \\ &= \frac{1}{\varepsilon} k\left(\sigma + \frac{\varepsilon}{p}\right) \left( \frac{1}{V_{n_0}^\varepsilon} + \varepsilon O(1) \right). \end{aligned}$$

If there exists a positive constant  $K \geq k(\sigma)$  such that (42) is valid when replacing  $k(\sigma)$  by  $K$ , then, in particular, we have  $\varepsilon \tilde{I} > \varepsilon K \|\tilde{f}\|_{p,\tilde{\Phi}_\delta} \|\tilde{a}\|_{q,\Psi}$ , namely,

$$k\left(\sigma + \frac{\varepsilon}{p}\right) \left( \frac{1}{V_{n_0}^\varepsilon} + \varepsilon O(1) \right) > K (U^{\delta\varepsilon}(1) - \varepsilon O(1))^{\frac{1}{p}} \left( \frac{1}{V_{n_0}^\varepsilon} + \varepsilon O(1) \right)^{\frac{1}{q}}.$$

It follows that  $k(\sigma) \geq K$  ( $\varepsilon \rightarrow 0^+$ ). Hence,  $K = k(\sigma)$  is the best possible constant factor of (47).

The constant factor  $k(\sigma)$  in (48) ((49)) is still the best possible. Otherwise, we would reach a contradiction by (50) ((51)) that the constant factor in (47) is not the best possible.  $\square$

## 5 Some corollaries

For  $\delta = 1$  in Theorems 2-4, we have the following inequalities with inhomogeneous kernel.

**Corollary 1** *If  $\gamma, \rho > 0$ ,  $\alpha > \rho$  ( $\alpha \geq 0$ ),  $0 < \sigma \leq 1$ ,  $k(\sigma)$  is indicated by (12), there exists  $n_0 \in \mathbf{N}$  such that  $v_n \geq v_{n+1}$  ( $n \in \{n_0, n_0 + 1, \dots\}$ ), and  $U(\infty) = V(\infty) = \infty$ , then*

(i) for  $p > 1$ ,  $0 < \|f\|_{p,\Phi_1}$ ,  $\|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n f(x) dx < k(\sigma) \|f\|_{p,\Phi_1} \|a\|_{q,\Psi}, \quad (52)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} f(x) dx \right]^p < k(\sigma) \|f\|_{p,\Phi_1}, \quad (53)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k(\sigma) \|a\|_{q,\Psi}; \quad (54)$$

(ii) for  $p < 0$ ,  $0 < \|f\|_{p,\Phi_1}$ , and  $\|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n f(x) dx > k(\sigma) \|f\|_{p,\Phi_1} \|a\|_{q,\Psi}, \quad (55)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} f(x) dx \right]^p > k(\sigma) \|f\|_{p,\Phi_1}, \quad (56)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) \|a\|_{q,\Psi}; \quad (57)$$

(iii) for  $0 < p < 1$ ,  $0 < \|f\|_{p,\Phi_1}$ , and  $\|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n f(x) dx > k(\sigma) \|f\|_{p,\tilde{\Phi}_1} \|a\|_{q,\Psi}, \quad (58)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} f(x) dx \right]^p > k(\sigma) \|f\|_{p,\tilde{\Phi}_1}, \quad (59)$$

$$\left\{ \int_0^{\infty} \frac{(1-\theta_1(\sigma,x))^{1-q} \mu(x)}{U^{1-q\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U(x)V_n)^{\gamma})}{e^{\alpha(U(x)V_n)^{\gamma}}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) \|a\|_{q,\Psi}. \quad (60)$$

The above inequalities are with the best possible constant factor  $k(\sigma)$ .

For  $\delta = -1$  in Theorems 2-4, we have the following inequalities with the homogeneous kernel of degree 0.

**Corollary 2** If  $\gamma, \rho > 0$ ,  $\alpha > -\rho$  ( $\alpha \geq 0$ ),  $0 < \sigma \leq 1$ ,  $k(\sigma)$  is defined in (12), there exists  $n_0 \in \mathbf{N}$  such that  $v_n \geq v_{n+1}$  ( $n \in \{n_0, n_0 + 1, \dots\}$ ), and  $U(\infty) = V(\infty) = \infty$ , then

(i) for  $p > 1$ ,  $0 < \|f\|_{p,\Phi_{-1}}$ , and  $\|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} a_n f(x) dx < k(\sigma) \|f\|_{p,\Phi_{-1}} \|a\|_{q,\Psi}, \quad (61)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(\frac{V_n}{U(x)})^{\gamma})}{e^{\alpha(\frac{V_n}{U(x)})^{\gamma}}} f(x) dx \right]^p < k(\sigma) \|f\|_{p,\Phi_{-1}}, \quad (62)$$

$$\left\{ \int_0^\infty \frac{\mu(x)}{U^{1+q\sigma}(x)} \left[ \sum_{n=1}^\infty \frac{\sec h(\rho(\frac{V_n}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n}{U(x)})^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < k(\sigma) \|a\|_{q,\Psi}; \quad (63)$$

(ii) for  $p < 0$ ,  $0 < \|f\|_{p,\Phi_{-1}}$ , and  $\|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\sec h(\rho(\frac{V_n}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n}{U(x)})^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p,\Phi_{-1}} \|a\|_{q,\Psi}, \quad (64)$$

$$\sum_{n=1}^\infty \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^\infty \frac{\sec h(\rho(\frac{V_n}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n}{U(x)})^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p,\Phi_{-1}}, \quad (65)$$

$$\left\{ \int_0^\infty \frac{\mu(x)}{U^{1+q\sigma}(x)} \left[ \sum_{n=1}^\infty \frac{\sec h(\rho(\frac{V_n}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n}{U(x)})^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) \|a\|_{q,\Psi}; \quad (66)$$

(iii) for  $0 < p < 1$ ,  $0 < \|f\|_{p,\Phi_{-1}}$ , and  $\|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities:

$$\sum_{n=1}^\infty \int_0^\infty \frac{\sec h(\rho(\frac{V_n}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n}{U(x)})^\gamma}} a_n f(x) dx > k(\sigma) \|f\|_{p,\tilde{\Phi}_{-1}} \|a\|_{q,\Psi}, \quad (67)$$

$$\sum_{n=1}^\infty \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^\infty \frac{\sec h(\rho(\frac{V_n}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n}{U(x)})^\gamma}} f(x) dx \right]^p > k(\sigma) \|f\|_{p,\tilde{\Phi}_{-1}}, \quad (68)$$

$$\left\{ \int_0^\infty \frac{(1 - \theta_{-1}(\sigma, x))^{1-q} \mu(x)}{U^{1+q\sigma}(x)} \left[ \sum_{n=1}^\infty \frac{\sec h(\rho(\frac{V_n}{U(x)})^\gamma)}{e^{\alpha(\frac{V_n}{U(x)})^\gamma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > k(\sigma) \|a\|_{q,\Psi}. \quad (69)$$

The above inequalities are with the best possible constant factor  $k(\sigma)$ .

For  $\alpha = \rho$  and  $\gamma = \sigma$  in Theorems 2-4, we have the following corollary.

**Corollary 3** If  $\rho > 0$ ,  $0 < \sigma \leq 1$ , there exists  $n_0 \in \mathbf{N}$  such that  $v_n \geq v_{n+1}$  ( $n \in \{n_0, n_0 + 1, \dots\}$ ), and  $U(\infty) = V(\infty) = \infty$ , then

(i) for  $p > 1$ ,  $0 < \|f\|_{p,\Phi_\delta}$ ,  $\|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\ln 2}{\sigma\rho}$ :

$$\sum_{n=1}^\infty \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} a_n f(x) dx < \frac{\ln 2}{\sigma\rho} \|f\|_{p,\Phi_\delta} \|a\|_{q,\Psi}, \quad (70)$$

$$\sum_{n=1}^\infty \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} f(x) dx \right]^p < \frac{\ln 2}{\sigma\rho} \|f\|_{p,\Phi_\delta}, \quad (71)$$

$$\left\{ \int_0^\infty \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^\infty \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} a_n \right]^q dx \right\}^{\frac{1}{q}} < \frac{\ln 2}{\sigma\rho} \|a\|_{q,\Psi}; \quad (72)$$

(ii) for  $p < 0$ ,  $0 < \|f\|_{p, \Phi_\delta}$ ,  $\|a\|_{q, \Psi} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\ln 2}{\sigma \rho}$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} a_n f(x) dx > \frac{\ln 2}{\sigma \rho} \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad (73)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} f(x) dx \right]^p > \frac{\ln 2}{\sigma \rho} \|f\|_{p, \Phi_\delta}, \quad (74)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} a_n \right]^q dx \right\}^{\frac{1}{q}} > \frac{\ln 2}{\sigma \rho} \|a\|_{q, \Psi}; \quad (75)$$

(iii) for  $0 < p < 1$ ,  $0 < \|f\|_{p, \Phi_\delta}$ , and  $\|a\|_{q, \Psi} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\ln 2}{\sigma \rho}$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} a_n f(x) dx > \frac{\ln 2}{\sigma \rho} \|f\|_{p, \tilde{\Phi}_\delta} \|a\|_{q, \Psi}, \quad (76)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} f(x) dx \right]^p > \frac{\ln 2}{\sigma \rho} \|f\|_{p, \tilde{\Phi}_\delta}, \quad (77)$$

$$\begin{aligned} & \left\{ \int_0^{\infty} \frac{(1 - \theta_\delta(\sigma, x))^{1-q} \mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \frac{\sec h(\rho(U^\delta(x)V_n)^\sigma)}{e^{\rho(U^\delta(x)V_n)^\sigma}} a_n \right]^q dx \right\}^{\frac{1}{q}} \\ & > \frac{\ln 2}{\sigma \rho} \|a\|_{q, \Psi}. \end{aligned} \quad (78)$$

For  $\alpha = 0$  and  $\gamma = \sigma$  in Theorems 2-4, we have the following corollary.

**Corollary 4** If  $\rho > 0$ ,  $0 < \sigma \leq 1$ , there exists  $n_0 \in \mathbb{N}$  such that  $v_n \geq v_{n+1}$  ( $n \in \{n_0, n_0 + 1, \dots\}$ ), and  $U(\infty) = V(\infty) = \infty$ , then

(i) for  $p > 1$ ,  $0 < \|f\|_{p, \Phi_\delta}$ , and  $\|a\|_{q, \Psi} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\pi}{2\sigma\rho}$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) a_n f(x) dx < \frac{\pi}{2\sigma\rho} \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad (79)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) f(x) dx \right]^p < \frac{\pi}{2\sigma\rho} \|f\|_{p, \Phi_\delta}, \quad (80)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) a_n \right]^q dx \right\}^{\frac{1}{q}} < \frac{\pi}{2\sigma\rho} \|a\|_{q, \Psi}; \quad (81)$$

(ii) for  $p < 0$ ,  $0 < \|f\|_{p, \Phi_\delta}$ , and  $\|a\|_{q, \Psi} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\pi}{2\sigma\rho}$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) a_n f(x) dx > \frac{\pi}{2\sigma\rho} \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad (82)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) f(x) dx \right]^p > \frac{\pi}{2\sigma\rho} \|f\|_{p,\Phi_\delta}, \quad (83)$$

$$\left\{ \int_0^{\infty} \frac{\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) a_n \right]^q dx \right\}^{\frac{1}{q}} > \frac{\pi}{2\sigma\rho} \|a\|_{q,\Psi}; \quad (84)$$

(iii) for  $0 < p < 1$ ,  $0 < \|f\|_{p,\Phi_\delta}$ , and  $\|a\|_{q,\Psi} < \infty$ , we have the following equivalent inequalities with the best possible constant factor  $\frac{\pi}{2\sigma\rho}$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) a_n f(x) dx > \frac{\pi}{2\sigma\rho} \|f\|_{p,\tilde{\Phi}_\delta} \|a\|_{q,\Psi}, \quad (85)$$

$$\sum_{n=1}^{\infty} \frac{v_n}{V_n^{1-p\sigma}} \left[ \int_0^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) f(x) dx \right]^p > \frac{\pi}{2\sigma\rho} \|f\|_{p,\tilde{\Phi}_\delta}, \quad (86)$$

$$\left\{ \int_0^{\infty} \frac{(1-\theta_\delta(\sigma,x))^{1-q}\mu(x)}{U^{1-q\delta\sigma}(x)} \left[ \sum_{n=1}^{\infty} \sec h(\rho(U^\delta(x)V_n)^\sigma) a_n \right]^q dx \right\}^{\frac{1}{q}} > \frac{\pi}{2\sigma\rho} \|a\|_{q,\Psi}. \quad (87)$$

**Remark 2** For  $\mu(x) = v_n = 1$  in (52), we have the following inequality with the best possible constant factor  $k(\sigma)$ :

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(x^\delta n)^\gamma)}{e^{\alpha(x^\delta n)^\gamma}} a_n f(x) dx \\ & < k(\sigma) \left[ \int_0^{\infty} x^{p(1-\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \end{aligned} \quad (88)$$

In particular, for  $\delta = 1$ , we have the following inequality with inhomogeneous kernel:

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(xn)^\gamma)}{e^{\alpha(xn)^\gamma}} a_n f(x) dx \\ & < k(\sigma) \left[ \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}; \end{aligned} \quad (89)$$

for  $\delta = -1$ , we have the following inequality with inhomogeneous kernel:

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\sec h(\rho(\frac{n}{x})^\gamma)}{e^{\alpha(\frac{n}{x})^\gamma}} a_n f(x) dx \\ & < k(\sigma) \left[ \int_0^{\infty} x^{p(1+\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{q(1-\sigma)-1} a_n^q \right]^{\frac{1}{q}}. \end{aligned} \quad (90)$$

We still can obtain a large number of other inequalities by using some particular parameters in theorems and corollaries.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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